A General Appro imation for the Di tri ution of Count Data

Edward Fox, Bezalel Gavish, John Semple*

Cox School of Business

Southern Methodist University

Dallas, TX - 75275

Abstract

Under mild assum tions about the interarrival distribution, we derive a modified version of the Birnbaum-Saunders distribution, which we call the tBISA, as an a roximation for the true distribution of count data. The free arameters of the tBISA are the first two moments of the underlying interarrival distribution. We show that the density for the sum of tBISA variables is available in closed form. This density is determined using the tBISA's moment generating function, which we introduce to the literature. The tBISA's moment generating function additionally reveals a new mixture inter retation that is based on the inverse Gaussian and gamma distributions. We then show that the tBISA can fit count data better than the distributions commonly used to model demand in economics and business. In numerical ex eriments and em irical a lications, we demonstrate that modeling demand with the tBISA can lead to better economic decisions.

Keywords: Birnbaum-Saunders; inverse Gaussian; gamma; confluent hy ergeometric functions; inventory model.

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Corres onding author: jsem le@cox.smu.edu

can still be obtained from interarrival data. We demonstrate both estimation approaches.

Because man applications require that counts be summed, e investigate the additive properties of the tBISA. For example, the interarrival distribution ma change (e.g., b time-of-da , da -of-the- eek or season), thereb violating the assumption that arrival times are identicall distributed. Another example involves d namic inventor models. Determining the optimal polic parameters in some d namic inventor models requires aggregating demand over the number of periods in the deliver lag.

Determining the sum of tBISA random variables requires that e derive the BISA's moment generating function (mgf), hich appears to have been previousl undiscovered (interestingl , the mgf of the log-BISA, also called the sinh-normal, is kno n, albeit in terms of modified Bessel functions of the third kind [15]). The BISA mgf reveals that the distribution can be represented as a mixture, in equal proportions, of (i) an inverse Gaussian and (ii) the same inverse Gaussian plus an independent gamma distribution ith shape 93 -1 .4d4[(is)-477(kno)27(n,)-5 central limit theorem, the probability of the count C being n or less is

$$\Pr(C \leq n) = \Pr\left(\sum_{i=1}^{n+1} X_i > T\right) = \Pr\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n+1}} > \frac{T/(n+1) - \mu}{\sigma/\sqrt{n+1}}\right)$$

$$\approx 1 - \Phi\left(\frac{T/(n+1) - \mu}{\sigma/\sqrt{n+1}}\right) \qquad (2)$$

where $\Phi(\,)$ is the cumulative distribution function for the standard normal. Approximating the discrete count n with a continuous variable $x \geq 0$, we obtain the density

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{[T - \mu(x+1)]}{\sigma\sqrt{x+1}}\right)^2\right) \frac{T + \mu(x+1)}{(x+1)^{3/2}}. \qquad (3)$$

By comparison, Birnbaum and Saunders [2] use n instead of $(n+1)$ when modeling the number of cycles until failure (this is because $n = 0$ is not a possibility in their model; it is in ours), so their density is

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{[T - \mu x]}{\sigma\sqrt{x}}\right)\right.$$ ET q 1 0 0 1 290.474 549.08 8.546 T

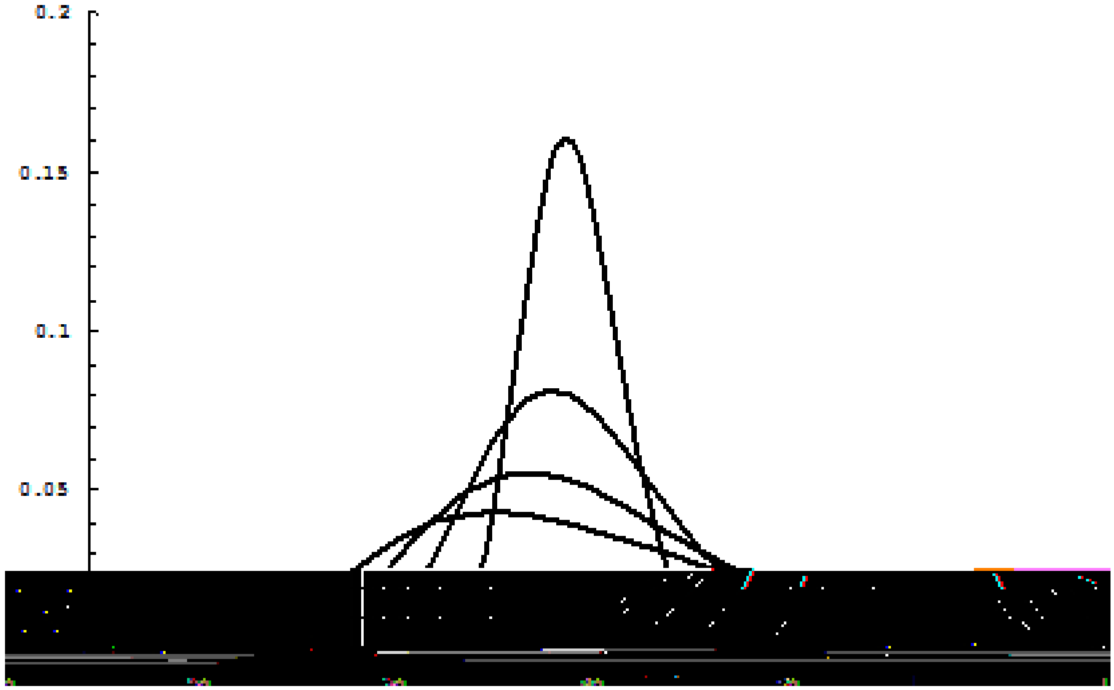

Figure 1: tBIS density for $T = 500$, $= 20$, $= 10, 20, 30, 40$

this approximation is ver good. Thus, hile the next proposition states approximate results, the results are nearl exact for practical purposes.

Proposition 1. *Let the me n nd st nd rd devi tion of the (st tion ry) inter rriv l distribution be nd , respectively. Then the rst three moments bout the me n for the count distribution (5) re*

(i) $E(C) = \frac{T}{} \; .5 + \frac{^2}{2\;^2}$

(ii) $E(C \quad E(C))^2 = \frac{5\;^4}{4\;^4} + \frac{T}{} \quad \frac{^2}{^2}$

(iii) $E(C \quad E(C))^3 = \frac{11\;^6}{2\;^6} + \frac{T}{} \quad \frac{^4}{^4}$

ot surprisingl , result (i) is $1{=}2$ unit less than the corresponding result in [2] hile result (ii) is identical. Result (iii) can be obtained from [9] after a little algebra. We note that the moment formulas in Proposition 1 are all functions of just t o fundamental quantities, the coefficient of variation of the interarrival distribution, $=$, and the ratio $T{=}$. Moreover, the moments are all increasing functions of these t o terms. In particular, the third moment about the mean is al a s positive so the count distribution is al a s positivel ske ed.

Proposition . *The density (5) is unimod l, nd its mode is less th n its medi n which is less th n its me n.*

When a tBIS random variable (5) is log-transformed, it produces a s mmetric, unimodal distribution that resembles a normal distribution. This result is analogous to that obtained in [1] for the BIS distribution (4).

Proposition . *Suppose th t the count* C *h s the density (5). Then* $Y = \ln(C + .5)$ *h s unimod l distribution th t is symmetric bout* $\ln(T=)$.

The proof of Proposition 3 is straightfor ard, and the proposition provides a theoretical basis for modeling the logarithm of count data, as is customaril done in man applications in economics and business. It is orth noting, ho ever, that the tBIS■ distribution retains an important advantage over logarithmic distributions—it is derived directl from the interarrival distribution hose moments define its free parameters.

3. SO E CO PA ISONS WITH EXACT COUNT DIST IBUTIONS

We no assess the accurac of the tBIS■ approximation. Under certain assumptions, the probabilit that the count C equals n can be computed exactl so a comparison bet een the tBIS■ distribution (5) and a kno n count distribution is possible. The primar requirements for the interarrival distribution are that (i) the interarrival distribution has nonnegative support and (ii) the distribution for the sum can be determined in a convenient numerical form. We consider t o such cases here. The first is a gamma interarrival process, hich nests the exponential, Erlang, and chi-square as special cases. The second is a uniform interarrival process. For comparing fits, e report the mean and variance of each distribution (exact count distribution vs. tBIS■) as ell as the maximum absolute value of the difference, D_{max}, bet een the cdf of the exact count distribution and the cdf of the tBIS■.

.1 *Ga a Interarri a*

We follo the development of Winkelmann [19]. The time bet een arrivals is gamma distributed ith shape parameter $k > 0$ and scale parameter > 0. The time interval is $[0, T]$. The mean and variance are k and $k\ ^2$, respectivel . The interarrival time has probabilit densit

$$f(\ ;k;\ ) = \frac{1}{\ ^k\ (k)}\ ^{k-1}\exp(\ \ =\ ) \quad for \quad > 0\, and\, k;\ \ 2\ \mathrm{R}^+ \qquad (\ )$$

Define

$$G(nk; T=\ ) = \frac{1}{\ (nk)}\int_0^{T=}u^{nk-1}\exp(\ u)du. \qquad (7)$$

The count distribution on the interval $[0, T]$ is

$$P\,(C = n) = G(kn; T=\ ) \quad G(k(n+1); T=\ ) \qquad (8)$$

for $n = 0, 1, 2, \ldots$.

Figure 2 illustrates the exact count distribution for $k = 1{=}2$

	Gamma Interarrivals			Uniform Interarrivals	
	$k = \frac{1}{2}$, = 40	$k = 1$, = 20	$k = 2$, = 10	$T = 5$	$T = 10$
$count$	25.5	25	24.75	9. 7	19. 0
$tBISA$	25.5	25	24.75	9. 7	19. 7
$count$	7.0533 8	5	3.5443 1	1.88	2. 02
$tBISA$	7.41 198	5.123475	3.579455	1.8 339	2. 0875
D_{max}	.037 2	.02 0	.01881	.0029	.0015

Table 1: tBISA approximation compared to exact count distributions

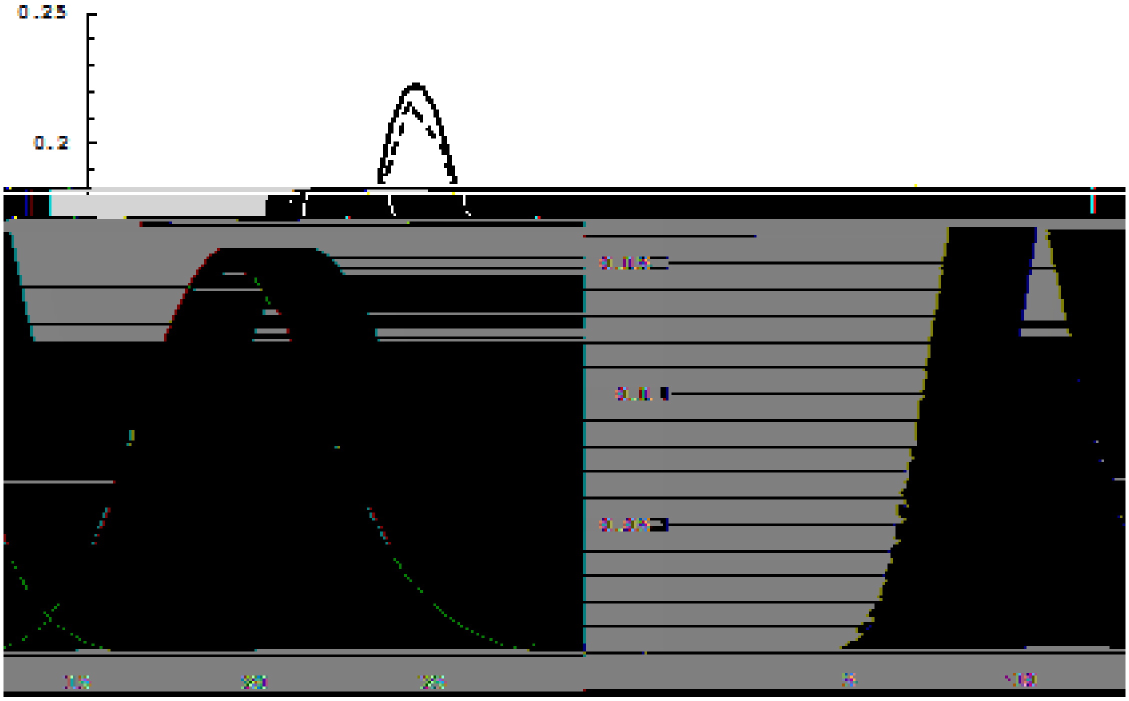

Figure 3: tBISA distribution (solid line) vs. exact count distribution (dashed line) assuming uniform interarrivals.

.2 *Unifor Interarri a*

Assume interarrival times are uniform $U[0,1]$. The mean and variance are 1/2 and 1/12, respectivel . Then the densit for $S_n = U_1 + U_2 + \cdots U_n$ is

$$f_n(x) = \frac{1}{2(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^{n-1} sgn(x-k) \quad 0 \le x \le n; \tag{9}$$

hich can be obtained after some algebra from Theorem 1 in [3]. From (9) one can compute the exact probabilit of the count equaling n for the time interval $[0,T]$

$$P(C=n) = P(S_{n+1} \ge T) - P(S_n \ge T) = \int_T^{n+1} [f_{n+1}(x) - f_n(x)]dx. \quad (T \le n+1) \tag{10}$$

Comparisons of the tBISA densit and $f_n(x)$ for T= 5, 10 are sho n in Figure 3 and their fits are compared in Table 1. In both cases, the tBISA approximates the exact count distribution extremel ell.

4. ADDITIVE P OPE TIES

In man applications, summing random counts is important. In economics and business applications, for example, the demand distribution ma var over time (e.g., b time-of-da or da -of-the- eek) so demand over the specified period can be represented as the sum of demands over disjoint subintervals. Also, man inventor problems require determining the distribution of demands

$$= \exp\left(\frac{T}{2}\left(1 - \sqrt{1 - 2t\frac{2}{2}}\right)\right) \quad for\ |t| < \frac{2}{2\ 2} \tag{13}$$

The mgf of the BIS distribution, $M_{BS}(t)$, can be expressed in terms of the mgf of the inverse Gaussian

$$\begin{aligned}
M_{BS}(t) &= \int_0^\infty \exp(tx)\frac{1}{2\sqrt{2}}\exp\left(-\frac{1}{2}\left(\frac{T-x}{\sqrt{x}}\right)^2\right)\frac{T+x}{x^{3/2}}\,dx \\
&= \frac{1}{2}\int_0^\infty \exp(tx)\frac{T}{\sqrt{2}}\exp\left(-\frac{1}{2}\left(\frac{T-x}{\sqrt{x}}\right)^2\right)\frac{1}{x^{3/2}}dx \\
&\quad + \frac{}{2T}\int_0^\infty \exp(tx)\frac{T}{\sqrt{2}}\exp\left(-\frac{1}{2}\left(\frac{T-x}{\sqrt{x}}\right)^2\right)\frac{1}{x^{1/2}}dx \\
&= \frac{1}{2}M_{IG}(t) + \frac{}{2T}M'_{IG}(t)
\end{aligned} \tag{14}$$

(Differentiation of $M_{IG}(t)$ in equation (14) can be justified for an $|t| < {}^2/2\ {}^2$ b appl ing Lebesgue's Dominated Convergence Theorem to the difference quotients.)

$$\begin{aligned}
&= \tfrac{1}{2}\exp\left(\tfrac{T}{2}\left(1-\sqrt{1-2t\tfrac{2}{2}}\right)\right) + \tfrac{}{2T}\exp\left(\tfrac{T}{2}\left(1-\sqrt{1-2t\tfrac{2}{2}}\right)\right)\left[\tfrac{T}{}\tfrac{1}{\sqrt{1-2t\frac{\sigma^2}{\mu^2}}}\right] \\
&= \tfrac{1}{2}\exp\left(\tfrac{T}{2}\left(1-\sqrt{1-2t\tfrac{2}{2}}\right)\right)\left[1+\tfrac{1}{\sqrt{1-2t\frac{\sigma^2}{\mu^2}}}\right] \quad \text{for } |t| < \tfrac{2}{2\ 2}
\end{aligned} \tag{15}$$

This establishes part (a). For part (b), the mgf in (a) can be ritten as

$$\frac{1}{2}\exp\left(\frac{T}{2}\left(1\right.\right.$$

04 -35.582 Td [(=)]TJ 19.86.8[(Td [(1)]TJ ET q 1 0 0 1 121.147 269.118 c

2 exp

hich characterized the BIS distribution as a mixture, in equal proportions, of an inverse Gaussian and a reciprocal inverse Gaussian (the distribution of $1/X$ here X inverse Gaussian). Moreover, our mixture interpretation allo s us to anal ze sums of independent BIS random variables having different parameters T_i, $_i$, and $_i$, something Desmond's interpretation does not facilitate. Finall , our mixture result implies that the reciprocal inverse Gaussian is equivalent to the sum of an inverse Gaussian and a gamma; this ill be revisited after Theorem 2.

Our discussion no turns to summing BIS random variables. The summation requires the use of *confluent hypergeometric functions*, hich are general solutions of the differential equation

$$z\frac{d^2w}{dz^2} + (b \quad z)\frac{dw}{dz} \quad aw = 0$$

introduced and anal zed b Kummer [12]. One solution is the

$$f_G(x) = \frac{1}{(k)\,(2 \;\; {}^2 = \;\; {}^2)^k} x^{k-1} \exp \quad \frac{{}^2 x}{2 \;\; {}^2} \; ! \; : \tag{20}$$

Therefore, the sum of the random variables has a densit given b the convolution $f_{IG+G}(s) = {}^{s}_{0} f$

$$g_k(s) = \int_0^\infty \exp\left(\frac{1}{2}\frac{T^2}{\ ^2}u \right) u^{k-1} \left(\frac{1}{us+1} \right)^{k-1/2} du = s^{-k} \int_0^\infty \exp\left(\frac{T^2}{2\ ^2}\frac{v}{s} \right) v^{k-1}(v+1)^{1/2-k} dv. \tag{2 }$$

For $z > 0$,

$$(a)U(a; b; z) = \int_0^\infty \exp(\ zt)\ t^{a-1}(1+t)^{b-a-1} dt; \tag{27}$$

(Formula 13.2.5 of 1, pg 505), hich for $a = 2$

Theorem . *Let* X_i *be r ndom v ri ble with BISA density (4) nd p r meters* T_i, $_i$, *nd* $_i$. *Assume* $_i$ *nd* $_i$ *dhere to property 1 nd the* X_i *re independent. Then* $\sum_{i=1} X_i$ *h s mixture distribution whose density is given by* $f(s) = (1=2)^n f_0(s) + \sum_{j=1} (1=2)^n \binom{n}{j} f_j(s)$ *where*

$$f_j(s) = \frac{T^{\ j}}{\sqrt{2}\ \ 2^{j=2}\ \ j+1} \exp\left(\frac{1}{2} \left(\frac{T}{\sqrt{s}} \ \ s \right)^2 \right) s^{j=2-3=2} U(j=2; =2; T^2=2\ ^2 s)$$

nd $= \sqrt{\sum_{i=1} {}_i^2}$, $= \sqrt{\ }$

Observe that the new parameters satisfy $\ = \ = v$ due to property 1. Then each term in the summation of (33) (ignoring the mixture weights) takes the general form

$$\exp\left(\frac{T}{\ ^2}\left(1-\sqrt{1-2tv^2}\right)\right)\left(1-2tv^2\right)^{-j/2}; \tag{35}$$

which is the mgf for the sum of (i) an inverse Gaussian with parameters $\ = T^2/\ ^2$ and $! = T/\ $ for T, and as defined in (34) and (ii) an independent gamma with shape parameter $j/2$ and scale parameter $\ = 2\ ^2/\ ^2 = 2v^2$. By Theorem 2, each of these has a density f_j involving the confluent hypergeometric function of the second kind,

$$f_0(s) = \frac{T}{\sqrt{2}}\exp\left(-\frac{1}{2}\left(\frac{T}{\sqrt{s}} - \frac{s}{\sqrt{s}}\right)^2\right)s^{-3/2} \quad for\ j = 0$$

$$f_j(s) = \frac{T^{\ j}}{\sqrt{2}\ ^{j+1}2^{j/2}}\exp\left(-\frac{1}{2}\left(\frac{T}{\sqrt{s}} - \frac{s}{\sqrt{s}}\right)^2\right)s^{j/2-3/2}U(j/2; 3/2; T^2/2\ ^2s) \quad for\ j = 1; 2; 3; \ldots$$

(3?)

The density for the sum of independent BISA random variables whose interarrival distributions have the same coefficient of variation is therefore the mixture

$$f(x) = (1/2)^n f_0(s) + \sum_{j=1}^{n}(1/2)^n\binom{n}{j}f_j(s). \tag{37}$$

This is a closed form representation involving confluent hypergeometric functions. □

Clearly, the shape of the final density in Theorem 3 is determined by the shape of the individual densities $f_j(x)$. To understand how T, , and affect the overall shape, we graphed the individual densities $j = 0, 1, 2, 3, 4, 5$ for two numerical cases: when $T = 500$, $= 20$, and $= 10$ (Figure 4); and when $T = 500$, $= 20$, and $= 40$ (Figure 5). Mixing the two leftmost densities in equal proportions (.5, .5) corresponds to the BISA distribution. Mixing the three leftmost densities in proportions (.25, .50, .25) corresponds to adding two BISA distributions. Mixing the four leftmost densities in proportions (.125, .375, .375, .125) corresponds to adding three BISA distributions, etc. As one might expect, the individual densities exhibit greater spread as the coefficient of variation increases from $v = .5$ (Figure 4) to $v = 2$ (Figure 5). Moreover, the expected values for the $f_j(s)$ increase with v as well. This result could be obtained directly by considering the expected value formula for a single BISA random variable (see Proposition 1).

Recall that the mgf for the tBISA introduces a factor $e^{-t/2}$ into the expression of Theorem 1, so the mgf for the sum of m such tBISAs includes an additional factor $e^{-mt/2}$. This amounts to shifting all of the mixture densities in Theorem 3 to the left by $m/2$ units. We also note that the parameters , , and T defined in Theorem 3 are not the only possible choices. These were chosen because they are easy to interpret. The proof of Theorem 3 goes through for other choices provided (i) $(\ /\ ) = v$ and (ii) $T/\ = \sum_{i=1} T_i/\ _i$. This implies that the density in Theorem 3 is governed by two unknown parameters provided the number of terms in the sum, n, is known. Alternatively, one could think of the parameter n as a third unkno n parameter in a generalized tBISA distribution.

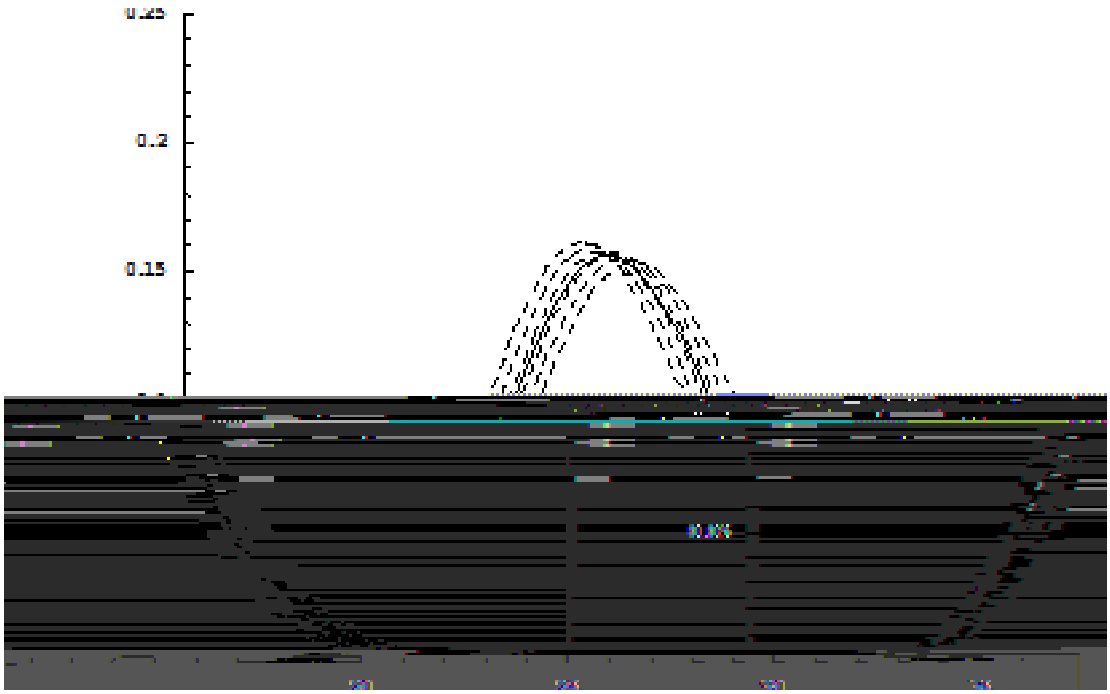

Figure 4: Mixture densities $f_j(s)$, $j = 0, 1, 2, 3, 4, 5$ (dashed lines); densit of sum $f(x)$(solid line) for $T = 500$, $= 20$, $= 10$.

. APPLICATIONS

5.1 *An E pirica Te t: Fitting the tBISA to De and Data*

Additional tests are required to determine the suitabilit of the tBISA as an approximation to the distribution of count data. Our testing ill focus on demand, the count of individual purchases, hich is commonl anal zed in economics and business problems. Accordingl , e use the term "interpurchase" as a more descriptive s non m for "interarrival" throughout this discussion. Our first test involved fitting the tBISA to actual demand data. We obtained demand data for the best-selling carbonated beverage at a local convenience store. Three hundred and eight -five da s of data ere available. We estimated the demand distribution using dail sales counts so that the input data as consistent across the candidate distributions e considered. It is interesting to note that the interpurchase distribution as not stationar over the entire da , so the assumptions under hich e derived the tBISA ere not, strictl speaking, met. This means the conditions for fitting the tBISA ere less than ideal.

The normal and lognormal distributions are most commonl used to fit demand data in practice. We therefore fit these t o distributions plus the Poisson and tBISA. All but the tBISA are easil fit using closed-form maximum likelihood estimates. The tBISA does not have closed form maximum likelihood estimates (these can be found via numerical optimization) but does have closed form

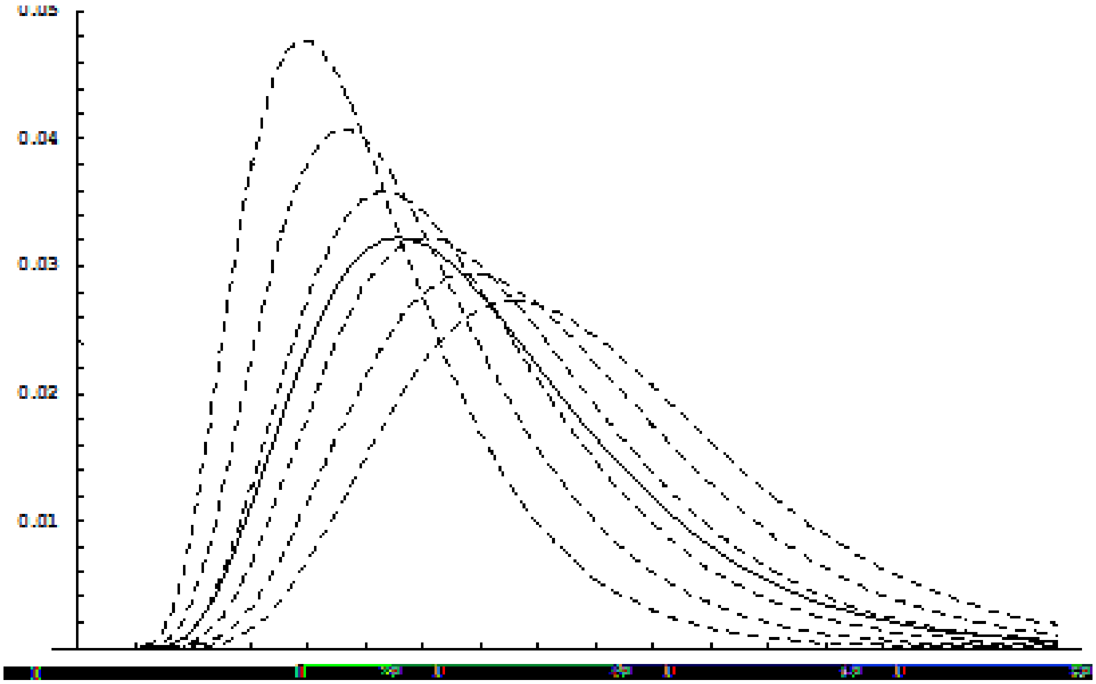

Figure 5: Mixture densities $f_j(s)$, $j = 0, 1, 2, 3, 4, 5$ (dashed lines); densit of sum $f(x)$(solid line) for $T = 500$, $= 20$, $= 40$.

method of moments estimates hich e use instead (see appendix). We computed D_{max} for each distribution as compared to the empirical demand distribution. We also computed D_{max} restricted to the top decile of the empirical distribution because the upper tail of the demand distribution is t picall most critical in business and economics applications. The results are summarized in Table 2, hich clearl sho s that the tBIS fits the carbonated beverage data better than the commonl used distributions. This is evident both for the entire distribution and for the upper tail.

	ormal	Lognormal	tBIS	Poisson
D_{max}	.075	.052	.042	.087
D_{max}				

here T is the time period, $_I$ is the mean interpurchase time, $_I$ is the standard deviation of the interpurchase time, and is the cdf for the standard normal distribution. The optimal Q therefore satisfies

$$[T \quad (Q+1/2)\ _I]/[\ _I \sqrt{Q+1/2}] = z_{1-} ; \tag{39}$$

here $z \quad = \quad ^{-1}(\ )$. Using a little algebra and the fact that $z_{1-} \quad = \quad z$, e determine that the optimal Q is

$$Q^* = T/\ _I \quad 1/2 + z^2(\ _I/\ _I)^2 + 1/2\sqrt{(z \quad _I/\ _I)^4 + 4(z \quad _I/\ _I)^2 T/\ _I}. \tag{40}$$

Observe that this quantit depends onl on parameters of the interpurchase distribution ($T/\ _I$, $_I/\ _I$) and the same critical value one ould use if the distribution of demand as assumed to be normal.

We applied the tBISA to the semiconductor demand data used b Gallego [5]. Sample statistics for eekl demand are $x_D = 207$ and $s_D^2 = 210681$. Assuming an overage cost of $h = \$2$ and a shortage cost of $s = \$5$, the optimal order quantit based on the empirical distribution of demand is approximatel 100 units, hich leads to an optimal profit of \$ 9. In contrast, the optimal order quantit based on a normal distribution leads to a loss of \$291. Gallego found the lognormal distribution as a much better alternative. Using the method of moments to fit a lognormal distribution to the demand data, he determined the optimal order quantit to be 181 ith a corresponding profit of \$29–a vast improvement over the normal distribution.

Distribution	Optimal Q	Optimal Profit
ormal	4 7	-\$291
Lognormal	181	\$29
tBISA	137	\$50.72
Empirical	100	\$ 9

Table 3: Comparison of optimal inventor levels and profits

Using the same data and cost assumptions, e found the tBISA distribution produced materiall better results. As Gallego did for the lognormal distribution, e used the method of moments (see appendix) to fit the tBISA. This results in estimates of $T/\ _I = 2.78525$ and $^2_I/\ ^2_I = 409.42949$ (note that these values are calculated from the demand data, not from interpurchase times). The optimal order quantit using these estimates is $Q^* = 137$ and the optimal profit is at least \$50.72 (this follo s from concavit of the profit function; e cannot be more precise ithout the full dataset hich is no longer available). The results are summarized in Table 3.

5. *App ication to Dyna ic In entory Mode*

The distribution of demand also pla s an essential role in more complicated models of inventor /production. In practice, the true distribution is t picall unkno n (see [ ]) so selecting a robust approximation is important. In some inventor /production applications, one must determine aggregate demand over multiple periods and so distributions that have additive properties are preferred. To determine if the tBIS■ holds promise in such settings, e conduct a simulation experiment using demand generated from a gamma interpurchase distribution. This interpurchase distribution as selected because it allo s for over-, under-, and equi-dispersion in the corresponding count (demand) distribution and because one can compute probabilities for the exact count distribution using the incomplete gamma (see equations 7 and 8).

The distribution of aggregate demand is a fundamental concern in d namic inventor models. In these models, one considers the short and long term costs of inventor over a multi-period horizon. T pical inventor costs include (i) the cost of ordering/purchasing inventor , (ii) the cost of holding excess inventor , and (iii) the the cost of either backlogging an item (if excess demand is backordered) or losing a sale (if excess demand is lost). In some d namic models, it is possible to describe in compact form the optimal order/purchase decision–other ise termed the *optim l policy* a single integral.

We considered three possible parameter combinations for gamma distributed interpurchases: $(k; \ )$ = (.5, 40), (1, 20), and (2, 10). Each combination implies a mean interarrival of 20; stan-

n	k		**Normal**	**Lognormal**	**tBISA-C**	**tBISA-I**
10	0.5	40	4.02	4.74	4.1	2.4
10	1	20	1.88	2.1		

N	k		Normal	Lognormal	tBISA-C	tBISA-I
10	0.5	40	5.9		5.88	4.92
10	1	20	3.4	3.4	3.	3.34
10	2	10	2.82	2.7	2. 2	2. 8
25	0.5	40	4.14	3.94	4.72	3.54
25	1	20	1.8	1.82	2.02	1.88
25	2	10	2.3	2.2	2.04	1.94
50	0.5	40	2.9	2.9	3.8	2.38
50	1	20	1.44	1.44	1.54	1.42
50	2	10	1.94	1.88	1.44	1.32
100	0.5	40	2.04	1.92	2.72	1.78
100	1	20	0.88	0.92	1.24	1.04
100	2	10	1.	1.5	1.3	0.92
200	0.5	40	1.72	1.	2.4	1.3
200	1	20	0.82	0.84	0.9	0.9
200	2	10	1.	1.54	0.9	0.5

third extension, to address nonstationarit in the interarrival distribution, ould be to partition the interarrivals into distinct groups or segments. For example, interarrivals times during different parts of the da (e.g., da time versus nighttime), different da s-of-the- eek (e.g., eekda versus eekend), or different seasons of the ear could be partitioned and their respective count distributions fit separatel . Alternativel , interarrival times could be separated based on a criterion that does not depend on time, e.g., cash customers versus credit customers (here e ould measure the time bet een cash purchases and the time bet een credit purchases). In each case, the total demand ould be the sum of counts for the different groups or segments. In other applications, the number of segments might not be kno n, in hich case n, the number of segments, becomes a free parameter in Theorem 3.

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$$\frac{T}{l} = \frac{(x_D + 1/2)}{3}\left(4 - \sqrt{1 + 3\frac{s_D^2}{(x_D + 1/2)^2}}\right) \qquad (44)$$

A limitation of this method is that it fails if $s_D^2/(x_D + 1/2)^2 \geq 5$, thus a different estimation method (e.g., maximum likelihood) would be required. Fortunately, this violation rarely occurs in practice, and so the method of moments should be broadly applicable.